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ON PHASE TRANSITIONS FOR P-ADIC POTTS MODEL WITH COMPETING INTERACTIONS ON A CAYLEY TREE

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Abstract

We consider a nearest-neighbor p-adic Potts (with $q \geq 2$ spin values and coupling constant $J \in \mathbb{Q}_p$) model on the Cayley tree of order $k \geq 1$. It is proved that a phase transition occurs at $k = 2$, $q \in p\mathbb{N}$ and $p \geq 3$ (resp. $q \in 2^2\mathbb{N}$, $p = 2$). It is established that for p-adic Potts model at $k \geq 3$ a phase transition may occur only at $q \in p\mathbb{N}$ if $p \geq 3$ and $q \in 2^2\mathbb{N}$ if $p = 2$.

Keywords: p-adic field, Potts model, Cayley tree, Gibbs measure, phase transition

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1 Introduction

The p-adic numbers were first introduced by the German mathematician K.Hensel. For about a century after the discovery of p-adic numbers, they were mainly considered objects of pure mathematics. However, numerous applications of these numbers to theoretical physics have been proposed papers [1-5] to quantum mechanics [6], to p-adic - valued physical observables [6] and many others [7,8]. A number of p-adic models in physics cannot be described using ordinary probability theory based on the Kolmogorov axioms [9]. New probability models - p-adic probability models were investigated in [8],[10].

In [11,12] a theory of stochastic processes with values in p-adic and more general non-Archimedean fields was developed , having probability distributions with non-Archimedean values.

One of the basic branches of mathematics lying at the base of the theory of statistical mechanics is the theory of probability and stochastic processes. Since the theories of probability and stochastic processes in a non-Archimedean setting

have been introduced, it is natural to study problems of statistical mechanics in the context of the p-adic theory of probability.

Many physical models are considered on $\mathbb{Z}^d$ and the Cayley tree. The difference between these two graphs is that the Cayley tree has the property that the number of neighbors visited in n steps grows exponentially with n. This is a faster rate of growth than n^d, no matter how large d is, so this tree is infinite dimensional.

In this paper we develop the p-adic probability theory approach to study some statistical mechanics models on a Cayley tree over the field of p-adic numbers. In [13] the Potts model with q spin variables on the set of integers $\mathbb{Z}$ in the field of $\mathbb{Q}_p$ was studied. It is known [14] that for the Potts model and even for arbitrary models on $\mathbb{Z}$ regardless of the interaction radius of the particles (over $\mathbb{R}$) there are no phase transitions, here the phase transition means that for given Hamiltomian there are at least two Gibbs measures. In the case considered in [13] this pattern is destroyed, namely, there are some values $q = q(p)$ for which phase transition occurs.

In the present paper we consider p-adic Potts models (with coupling constant J and q spin variables) on the Cayley tree of order k, $k \geq 1$. The aim of this paper is to investigate Gibbs measures for p-adic Potts model and phase transition problem for this model. The organization of this paper as follows.

Section 2 is a mathematically preliminary. In section 3 we give a construction of Gibbs measure for the p-adic Potts model on the Cayley tree. In section 4 we prove the existence of a phase transition for p-adic Potts model on Cayley tree of order two. In final section 5 we exhibit some conditions implying q on uniqueness of the Gibbs measure.

2 Definitions and preliminary results

2.1 p-adic numbers and measures

Let $\mathbb{Q}$ be the field of rational numbers. Every rational number $x \neq 0$ can be represented in the form $x = p^r \dfrac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, $(p, n) = 1$, $(p, m) = 1$ and p is a fixed prime number. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

This norm satisfies so called the strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

this is a non-Archimedean norm.

The completion of $\mathbb{Q}$ with respect to p-adic norm defines the p-adic field which is denoted by $\mathbb{Q}_p$. Any p-adic number $x \neq 0$ can be uniquely represented

in the form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...), \qquad (2.1)$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \le x_j \le p-1$, $x_0 > 0$, $j = 0, 1, 2, ...$ (see more detail [7,15]). In this case $|x|_p = p^{-\gamma(x)}$.

We recall that an integer $a \in \mathbb{Z}$ is called *a quadratic residue modulo* p if the equation $x^2 \equiv a (\text{mod } p)$ has a solution $x \in \mathbb{Z}$.

Lemma 2.1.[7] *In order that the equation*

$$x^2 = a, \quad 0 \neq a = p^{\gamma(a)}(a_0 + a_1 p + ...), \quad 0 \le a_j \le p-1, \ a_0 > 0$$

has a solution $x \in \mathbb{Q}_p$, it is necessary and sufficient that the following conditions are fulfilled:

i) $\gamma(a)$ is even;

ii) a_0 is a quadratic residue modulo p if $p \neq 2$, $a_1 = a_2 = 0$ if $p = 2$.

Let $B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}$, where $a \in \mathbb{Q}_p$, $r > 0$. By $\log_p$ and $\exp_p$ we mean p-adic logarithm and exponential which are defined as series with the usual way (see, for more details [7]. The domain of converge for them are $B(1, 1)$ and $B(0, p^{-1/(p-1)})$ respectively.)

Lemma 2.2.[7,15,16] *Let $x \in B(0, p^{-1/(p-1)})$ then we have*

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p < 1, \quad |\log_p(1 + x)|_p = |x|_p < p^{-1/(p-1)}$$

and

$$\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1 + x)) = 1 + x.$$

Let $(X, \mathcal{B})$ be a space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a *p-adic measure* if for any $A_1, ..., A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ $(i \neq j)$

$$\mu\Big(\bigcup_{j=1}^{n} A_j\Big) = \sum_{j=1}^{n} \mu(A_j).$$

A p-adic measure is called *a probability measure* if $\mu(X) = 1$. A p-adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$.

For more detail information about p-adic measures we refer to [8],[10].

2.2 The Cayley tree

The Cayley tree Γ^k of order $k \ge 1$ is an infinite tree, i.e., a graph without cycles, such that each vertex of which lies on $k + 1$ edges . Let $\Gamma^k = (V, \Lambda)$, where V is the set of vertices of Γ^k, Λ is the set of edges of Γ^k. The vertices x and y are called *nearest neighbor*, which is denoted by $l = <x, y>$ if there exists an edge connecting them. A collection of the pairs $<x, x_1>, ..., <x_{d-1}, y>$ is called

a path from x to y. The distance $d(x,y), x,y \in V$ is the length of the shortest path from x to y in V.

We set

$$W_n = \{x \in V | d(x, x^0) = n\},$$

$$V_n = \cup_{m=1}^{n} W_m = \{x \in V | d(x, x^0) \leq n\},$$

$$L_n = \{l =< x, y >\in L | x, y \in V_n\},$$

for a fixed point $x^0 \in V$.

We write $x < y$ if the path from x^0 to y goes through x. Call vertex y a *direct successor* of x if $y > x$ and x, y are nearest neighbors. Denote by $S(x)$ the set of direct successors, i.e.

$$S(x) = \{y \in W_{n+1} : d(x,y) = 1\} \;\; x \in W_n,$$

Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k+1$.

2.3 The p-adic Potts model

Let $\mathbb{Q}_p$ be the field of p-adic numbers. By $\mathbb{Q}_p^{q-1}$ we denote $\underbrace{\mathbb{Q}_p \times ... \times \mathbb{Q}_p}_{q-1}$. The norm $\|x\|_p$ of an element $x \in \mathbb{Q}_p^{q-1}$ is defined by $\|x\|_p = \max\limits_{1 \leq i \leq q-1} \{|x_i|_p\}$, here $x = (x_1, ..., x_{q-1})$. By xy we mean the bilinear form on $\mathbb{Q}_p^{q-1}$ defined by

$$xy = \sum_{i=1}^{q-1} x_i y_i, \quad x = (x_1, \cdots, x_{q-1}), y = (y_1, \cdots, y_{q-1}).$$

Let $\Psi = \{\sigma_1, \sigma_2, ..., \sigma_q\}$, where $\sigma_1, \sigma_2, ..., \sigma_q$ are elements of $\mathbb{Q}_p^{q-1}$ such that $\|\sigma_i\|_p = 1$, $i = 1, 2, ..., q$ and

$$\sigma_i \sigma_j = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j \end{cases} \;\; (i, j = 1, 2, ..., q-1), \;\; \sigma_q = \sum_{i=1}^{q-1} \sigma_i. \tag{2.2}$$

Let $h \in \mathbb{Q}_p^{q-1}$, then we have $h = \sum_{i=1}^{q-1} h_i \sigma_i$ and

$$h\sigma_i = \begin{cases} h_i, & \text{for } i = 1, 2, ..., q-1, \\ \sum_{i=1}^{q-1} h_i, & \text{for } i = q \end{cases} \tag{2.3}$$

We consider the p-adic Potts model where spin takes values in the set Ψ. Write $\emptyset_n = \Psi^{V_n}$, this is the configuration space on V_n. The Hamiltonian $H_n : \emptyset_n \to \mathbb{Q}_p$ of the p-adic Potts model has the form

$$H_n(\sigma) = -J \sum_{<x,y> \in L_n} \delta_{\sigma(x), \sigma(y)}, \quad n \in \mathbb{N}, \tag{2.4}$$

here $\sigma = \{\sigma(x) : x \in V_n\} \in \emptyset_n$, $|J|_p < p^{-1/(p-1)}$, $J \neq 0$ and as before p is a fixed prime number. Here δ is the Kronecker symbol.

3 Construction of Gibbs measures

In this section we give a construction of a special class of Gibbs measures for p-adic Potts models on the Cayley tree.

To define Gibbs measure we need in the following

Lemma 3.1. *Let $h_x, x \in V$ be a $\mathbb{Q}_p^{q-1}$-valued function such that $\|h_x\|_p \in B(0, p^{-1/(p-1)})$ for all $x \in V$ and $J \in B(0, p^{-1/(p-1)})$. Then*

$$H_n(\sigma) + \sum_{x \in W_n} h_x \sigma(x) \in B(0, p^{-1/(p-1)})$$

for any $n \in \mathbb{N}$.

The proof easily follows from the strong triangle inequality for the norm $|\cdot|_p$.

Let $h : x \in V \to h_x \in \mathbb{Q}_p^{q-1}$ be a function of $x \in V$ such that $\|h_x\|_p < p^{-1/(p-1)}$ for all $x \in V$. Given $n = 1, 2, ...$ consider a p-adic probability measure $\mu^{(n)}$ on Ψ^{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp_p\{-H_n(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\}, \tag{3.1}$$

Here, as before, $\sigma_n : x \in V_n \to \sigma_n(x)$ and Z_n is the corresponding partition function:

$$Z_n = \sum_{\tilde{\sigma}_n \in \Omega_{V_n}} \exp_p\{-H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_x \tilde{\sigma}(x)\}.$$

Note that according to Lemma 3.1 the measures $\mu^{(n)}$ exist.

The compatibility condition for $\mu^{(n)}(\sigma_n), n \geq 1$ are given by the equality

$$\sum_{\sigma^{(n)}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \mu^{(n-1)}(\sigma_{n-1}), \tag{3.2}$$

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$.

We note that an analog of the Kolmogorov extension theorem for distributions can be proved for p-adic distributions given by (3.1) (see [12]). According to this theorem there exists a unique p-adic measure μ_h on $\emptyset = \Psi^V$ such that for every $n = 1, 2, ...$ and $\sigma_n \in \Psi^{V_n}$

$$\mu\bigg(\{\sigma|_{V_n} = \sigma_n\}\bigg) = \mu^{(n)}(\sigma_n).$$

μ_h will be called *p-adic Gibbs measure* for this Potts model. It is clear that the measure μ_h depends on function h_x. If the Gibbs measure for a given Hamiltonian is non unique then we say that for this model there is *a phase transition*.

The following statement describes conditions on h_x guaranteeing the compatibility condition of measures $\mu^{(n)}(\sigma_n)$.

Theorem 3.2. *The measures $\mu^{(n)}(\sigma_n)$, $n = 1, 2, ...$ satisfy the compatibility condition (3.2) if and only if for any $x \in V$ the following equation holds:*

$$h_x = \sum_{y \in S(x)} F(h_y; \theta, q) \tag{3.3}$$

here and below $\theta = \exp_p(J)$ *and the function* $F : \mathbb{Q}_p^{q-1} \to \mathbb{Q}_p^{q-1}$ *function is defined by* $F(h;\theta,q) = (F_1(h;\theta,q), ..., F_q(h;\theta,q))$ *with*

$$F_i(h;\theta,q) = \sum_{j=1, j\neq i}^{q-1} G_j(h';\theta,q) \quad i = 1,...,q-1,$$

where $h = (h_1, ..., h_{q-1}),\ h' = (h'_1, ..., h'_{q-1})\ h'_i = \sum_{j=1, j\neq i}^{q-1} h_j,\ i = 1,...,q-1,$

$$G_i(h_1,...,h_{q-1};\theta,q) = \log_p \left[\frac{(\theta-1)\exp_p(h_i) + \sum_{j=1}^{q-1}\exp_p(h_j) + 1}{\sum_{j=1}^{q-1}\exp_p(h_j) + \theta}\right],$$

$i = 1,...,q-1.$

Proof. *Necessity.* According to the compatibility condition (3.2) we have

$$Z_n^{-1}\sum_{\sigma^{(n)}}\exp_p\left[J\sum_{<x,y>\in L_n}\delta_{\sigma(x),\sigma(y)} + \sum_{x\in W_n} h_x\sigma(x)\right] =$$

$$Z_{n-1}^{-1}\exp_p\left[J\sum_{<x,y>\in L_{n-1}}\delta_{\sigma(x),\sigma(y)} + \sum_{x\in W_{n-1}} h_x\sigma(x)\right]. \tag{3.4}$$

It yields

$$\frac{Z_{n-1}}{Z_n}\sum_{\sigma(n)}\exp_p\left[J\sum_{x\in W_{n-1}}\sum_{y\in S(x)}\delta_{\sigma(x),\sigma(y)} + \sum_{x\in W_{n-1}}\sum_{y\in S(x)} h_y\sigma(y)\right] =$$

$$\prod_{x\in W_{n-1}}\exp_p\Big(h_x\sigma(x)\Big). \tag{3.5}$$

From this equality we find

$$\frac{Z_{n-1}}{Z_n}\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\sum_{\sigma(y)\in\Psi}\exp_p\Big(J\delta_{\sigma(x),\sigma(y)} + h_y\sigma(y)\Big) = \prod_{x\in W_{n-1}}\exp_p\Big(h_x\sigma(x)\Big). \tag{3.6}$$

Now fix $x \in W_{n-1}$. Dividing the equalities (3.6) with $\sigma(x) = \sigma_i$ and with $\sigma(x) = \sigma_q$ we obtain

$$\prod_{y\in S(x)}\frac{\sum_{\sigma(y)\in\Psi}\exp_p\Big(J\delta_{\sigma_i,\sigma(y)} + h_y\sigma(y)\Big)}{\sum_{\sigma(y)\in\Psi}\exp_p\Big(J\delta_{\sigma_q,\sigma(y)} + h_x\sigma(y)\Big)} = \exp_p\Big(h_x(\sigma_i - \sigma_q)\Big). \tag{3.7}$$

Using (2.3) the last equality can be rewritten as

$$\prod_{y\in S(x)}\frac{\sum_{m=1}^{q-1}\exp_p\Big(\sum_{j=1,j\neq m}^{q-1} h_x^{(j)}\Big) + (\theta-1)\exp_p\Big(\sum_{j=1,j\neq i}^{q-1} h_x^{(j)}\Big) + 1}{\sum_{m=1}^{q-1}\exp_p\Big(\sum_{j=1,j\neq m}^{q-1} h_x^{(j)}\Big) + \theta} =$$

$$\exp_p\left(\sum_{j=1, j\neq i}^{q-1} h_x^{(j)}\right), \tag{3.8}$$

here we have used the notation $h_x = (h_x^{(1)}, \cdots, h_x^{(q-1)})$. Writing $h_x^{(i)'} = \sum_{j=1,j\neq i}^{q-1} h_x^{(j)}$ we immediately get (3.3) from (3.8).

Sufficiency. Now assume that (3.3) is valid, then it implies (3.8), and hence (3.7). From (3.7) we obtain the following equality

$$a(x)\exp_p\left(h_x\sigma_i\right) = \prod_{y\in S(x)}\sum_{\sigma(y)\in\Psi}\exp_p\left(J\delta_{\sigma_i,\sigma(y)} + h_y\sigma(y)\right), \quad i = 1, 2, ..., q.$$

This equality implies

$$\prod_{x\in W_{n-1}} a(x)\exp_p\left(h_x\sigma(x)\right) =$$

$$\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\sum_{\sigma(y)\in\Psi}\exp_p\left(J\delta_{\sigma(x),\sigma(y)} + h_y\sigma(y)\right), \quad i = 1, 2, ..., q, \tag{3.9}$$

where

$$\sigma(z) = \begin{cases} \sigma_i, & z = x \\ \sigma(z), & z \neq x \end{cases} \qquad i = \overline{1, q}.$$

Writing $A_n(x) = \prod_{x\in W_n} a(x)$ we find from (3.9) and (3.2)

$$Z_{n-1}A_{n-1}\mu^{(n-1)}(\sigma_{n-1}) = Z_n\sum_{\sigma^{(n)}}\mu^{(n)}(\sigma_{n-1}, \sigma^{(n)})$$

Since each $\mu^{(n)}$, $n \geq 1$ is a p-adic probability measure, we have

$$\sum_{\sigma_{n-1}}\sum_{\sigma^{(n)}}\mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = 1, \quad \sum_{\sigma_{n-1}}\mu^{(n-1)}(\sigma_{n-1}) = 1.$$

Therefore from these equalities we find $Z_{n-1}A_{n-1} = Z_n$ which means that (3.2) holds.

Observe that according to this Theorem the problem of describing of p-adic Gibbs measures reduces to the describing of solutions of the equation (3.3).

4 The problem of phase transitions

4.1 The existence of phase transition for the p-adic Potts model

Write

$$\Lambda = \{h = (h_x \in \mathbb{Q}_p^{q-1}, x \in V) : h_x \text{ satisfies the equation (3.3)}\}.$$

To prove the existence of phase transition it suffices to show that there are two different sets of vectors in Λ. The description of arbitrary elements of the set Λ is a complicated problem.

In this paper we restrict ourselves to the description of translation - invariant elements of Λ, in which $h_x = h$ is independent of x.

Let $h_x = h = (h_1, ..., h_{q-1})$ for all $x \in V$. Then (3.3) implies

$$\exp_p(h_i) = \left(\frac{(\theta - 1)\exp_p(h_i) + \sum_{j=1}^{q-1} \exp_p(h_j) + 1}{\sum_{j=1}^{q-1} \exp_p(h_j) + \theta} \right)^k, \quad i = 1, 2, ..., q-1. \quad (4.1)$$

Observe that for every $i = 1, 2, ..., q-1$ $h_i = 0$ satisfies i-th equation. Substituting $h_j = 0$ at $j = 2, 3, ..., q-1$ and writing $z = \exp_p(h_1)$ from the first equation of (4.1)

$$z = \left(\frac{\theta z + q - 1}{z + \theta + q - 2} \right)^k. \quad (4.2)$$

Put

$$A = \theta z + q - 1, \quad B = z + \theta + q - 2.$$

Then by (4.2)

$$B^k (z - 1) = (\theta - 1)(z - 1)(A^{k-1} + A^{k-2}B + ... + B^{k-1}).$$

Hence if $z \neq 1$

$$B^k = (\theta - 1)(A^{k-1} + A^{k-2}B + ... + B^{k-1}). \quad (4.3)$$

Using Lemma 2.2 it is easy to see that

$$|A|_p \begin{cases} \leq \frac{1}{p}, & \text{if } q \in p\mathbb{N}, \\ = 1, & \text{if } q \notin p\mathbb{N}, \end{cases} \qquad p \geq 3,$$

$$|B|_p \begin{cases} \leq \frac{1}{p}, & \text{if } q \in p\mathbb{N}, \\ = 1, & \text{if } q \notin p\mathbb{N}, \end{cases} \qquad p \geq 3,$$

$$|A|_2 \begin{cases} \leq \frac{1}{4}, & \text{if } q \in 2^2\mathbb{N}, \\ = \frac{1}{2}, & \text{if } q \in 2\mathbb{N} \setminus 2^2\mathbb{N}, \\ = 1, & \text{if } q \notin 2\mathbb{N}, \end{cases} \qquad p = 2.$$

$$|B|_2 \begin{cases} \leq \frac{1}{4}, & \text{if } q \in 2^2\mathbb{N}, \\ = \frac{1}{2}, & \text{if } q \in 2\mathbb{N} \setminus 2^2\mathbb{N}, \\ = 1, & \text{if } q \notin 2\mathbb{N}, \end{cases} \qquad p = 2.$$

From these inequalities we get

1) If $p = 2$; $q \in 2\mathbb{N} \setminus 2^2\mathbb{N}$ we have

$$|B|_2 = |A|_2 = \frac{1}{2}, \quad |B|_2^k = \frac{1}{2^k} > \frac{1}{4} \cdot \frac{1}{2^{k-1}}$$
$$|\theta - 1|_2 |A^{k-1} + ... + B^{k-1}|_2 \leq \frac{1}{4} \cdot \frac{1}{2^{k-1}}.$$

Here we have used the strong triangle inequality and $|\theta-1|_2\leq\dfrac{1}{4}$ (see Lemma 2.2). From the last inequalities we infer that the equation (4.3) has no solution.

2) If $p\geq 3$, $q\notin p\mathbb{N}$ (resp. $q\notin 2\mathbb{N}$ if $p=2$) then

$$|B|_p=|A|_p=1,\quad |\theta-1|_p|A^{k-1}+...+B^{k-1}|_p\leq\frac{1}{p}.$$

Hence in this case the equation (4.3) has no solution either.

3) If $p\geq 3$, $q\in p\mathbb{N}$ (resp. $p=2$, $q\in 2^2\mathbb{N}$) then it is easy to see that the equation (4.3) may have a solution.

Thus we have proved the following

Theorem 4.1. *If $p\geq 3$, $q\in p\mathbb{N}$ (resp. $p=2$, $q\in 2^2\mathbb{N}$) then the equation (4.1) may has at least two solutions for every $k\geq 1$.*

According to Theorem 4.1 in the sequel we will assume that $p\geq 3$, $q\in p\mathbb{N}$ (resp. $p=2$, $q\in 2^2\mathbb{N}$). Note, that in case of $k>2$ the problem of description of the solutions of (4.3) becomes difficult. For simplicity we restrict ourselves to the case $k=2$. Then (4.3) has the form

$$z^2+(2\theta-\theta^2+2q-3)z+(q-1)^2=0. \tag{4.4}$$

Observe that the solution of (4.4) can be written by

$$z_{1,2}=\frac{-(2\theta-\theta^2+2q-3)\pm(\theta-1)\sqrt{\theta^2-2\theta+5-4q}}{2}. \tag{4.5}$$

We must check the existence of $\sqrt{\theta^2-2\theta+5-4q}$ and additionally the inequality $|z_{1,2}-1|_p<p^{-1/(p-1)}$ which is equivalent to the condition $|h_x|_p<p^{-1/(p-1)}$.

From $|2q-(\theta-1)^2|_p<p^{-1/(p-1)}$ we find $|\theta-1|_p<p^{-1/(p-1)}$ for every prime number p. It then follows from (4.5) that $|z_{1,2}-1|_p<p^{-1/(p-1)}$.

Now we check the existence of $\sqrt{(\theta-1)^2+4(1-q)}$. We use canonical form of p-adic numbers (2.1).

1) Let $p=2$. Then $|\theta-1|_2=|J|_2\leq\dfrac{1}{4}$, hence $\theta-1=2^\gamma\varepsilon$, $\gamma\geq 2$, $|\varepsilon|_2=1$. According to our assumption $q=2^{2+m}s$, $m\geq 0$, $(s,2)=1$, whence we can write $s=\sum_{i=0}^{l}c_i2^i$, $c_0=1$, $c_i\in\{0,1\}, i=1,2,...,l$. Hence we have

$$(\theta-1)^2+4(1-q)=2^2-2^{4+m}s+2^{2\gamma}\varepsilon^2=$$

$$=2^2(1+\sum_{i=0}^{l}(2-c_i)2^{2+m+i}+2^{2\gamma-2}\varepsilon^2). \tag{4.6}$$

2) Let $p=3$. Then $q=3^ms$, $(s,3)=1, m\geq 1$, whence $4s=\sum_{i=0}^{l}b_i3^i$, $b_0=1,2$, $b_i\in\{0,1,2\}, i=\overline{1,l}$. The inequality $|\theta-1|_3\leq\dfrac{1}{3}$ implies that $\theta-1=3^\gamma\varepsilon$, $\gamma\geq 1$, $|\varepsilon|_3=1$. Hence we get

$$(\theta-1)^2+4(1-q)=1+3+\sum_{i=0}^{l}(3-b_i)3^{m+i}+3^{2\gamma}\varepsilon^2.$$

3) Let $p \geq 5$. Then $q = p^m s$, $(s,p) = 1, m \geq 1$, whence $4s = \sum_{i=0}^{l} b_i p^i$, $b_0 = 1, 2, ..., p-1$, $b_i = 0, 1, ..., p-1$, $i = 1, 2, ..., l$. The inequality $|\theta - 1|_p \leq \frac{1}{p}$ implies that $\theta - 1 = p^\gamma \varepsilon$, $\gamma \geq 1$, $|\varepsilon|_p = 1$. Hence we get

$$(\theta-1)^2 + 4(1-q) = 4 + \sum_{i=0}^{l} (p-b_i)p^{m+i} + p^{2\gamma}\varepsilon^2.$$

We can now check all conditions of Lemma 2.1: observe that the each case the first condition of Lemma 2.1 is fulfilled. It remains to check the second condition, i.e. the equation $x^2 \equiv a_0 (\mathrm{mod}\ p)$ has solution $x \in \mathbb{Z}$.

1) Let $p = 2$. In this case $a_0 = 1$, then it is easy to see that the equation $x^2 \equiv 1 (\mathrm{mod}\ 2)$ has solution $x = 2N+1$, $N \in \mathbb{Z}$. Besides it must be $a_1 = a_2 = 0$. From (4.6) one can find that

$$a_1 = a_2 = 0 \ \text{ if and only if either } m = 0, \gamma = 2 \ \text{ or } m > 1, \gamma > 2.$$

Thus in this case for 2-adic Potts model a phase transition occurs.

2) Let $p = 3$. In this case $a_0 = 1$, then it is not difficult to check that the equation $x^2 \equiv 1 (\mathrm{mod}\ 3)$ has the solution $x = 3N + 1$, $N \in \mathbb{Z}$.

3) Let $p \geq 5$. In this case $a_0 = 4$, then $x^2 \equiv 2 (\mathrm{mod}\ p)$ has the solution $x = pN + 2$, $N \in \mathbb{Z}$.

Consequently we have proved the following

Theorem 4.2. *i) Let $p = 2, q \in 2^2\mathbb{N}$ and $J \neq 0$. If $q = 2^2 s$, $(s,2) = 1$, $|J|_2 = \frac{1}{4}$ or $q = 2^m s, m \geq 3, (s,2) = 1$, $|J|_2 \leq \frac{1}{4}$ then there exists a phase transition for the 2-adic Potts model (2.4) on a Cayley tree of order 2.*

ii) Let $p \geq 3$, $q \in p\mathbb{N}$, and $0 < |J|_p \leq \frac{1}{p}$ then there exists a phase transition for p-adic Potts model on a Cayley tree of order 2.

Observe that if $q = 2$ then the Potts model becomes the Ising model, so from this theorem and Theorem 4.1 we have the following

Corollary 4.3. *Let $k \geq 1$. Then for the p-adic Ising model on the Cayley tree of order k there is no phase transition.*

Conjecture. *Let all conditions of Theorem 4.2 be satisfied. Then there is a phase transition for the p-adic Potts model on the Cayley tree of order k ($k \geq 3$).*

Now we investigate when the p-adic Gibbs measure with the solutions (4.2) is bounded.

Theorem 4.4. *The p-adic Gibbs measure μ for the p-adic Potts model on the Cayley tree of order k is bounded if $q \notin p\mathbb{N}$, otherwise it is not bounded.*

Proof. To prove the assertion of theorem it suffices to show that the values of μ on cylindrical subsets are bounded. We estimate $|\mu^{(n)}(\sigma_n)|_p$:

$$|\mu^{(n)}(\sigma_n)|_p = \left| \frac{\exp_p\{\tilde{H}(\sigma_n)\}}{\sum_{\tilde{\sigma}_n \in \varnothing_{V_n}} \exp_p\{\tilde{H}(\sigma_n)\}} \right|_p =$$

$$\frac{1}{\left|\sum_{\tilde{\sigma}_n\in \O_{V_n}}(\exp_p\{\tilde{H}(\sigma_n)\}-1)+q^{V_n}\right|_p}=1$$

if $q\notin p\mathbb{N}$. Here

$$\tilde{H}(\sigma_n)=H(\sigma_n)+\sum_{x\in W_n}h_*\sigma(x),$$

and h_* is a solution of (4.4), and we have used $|\exp_p\{\tilde{H}(\sigma_n)\}-1|_p\leq \frac{1}{p}$. Note that h_* is a vector which has the form $h_*=(\underbrace{h,0,...,0}_{q-1})$.

Write

$$p_{ij}=\frac{\exp_p(J\delta_{ij}+h_*(i+j))}{\sum_{km}\exp_p(J\delta_{km}+h_*(k+m))},$$

where $i,j\in\{\sigma_1,...,\sigma_{q-1}\}$.

To prove that the measure μ is not bounded at $q\in p\mathbb{N}$ it is enough to show that its marginal (formdary) measure is not bounded. Let $\pi=\{...,x_{-1},x_0,x_1,...\}$ be an arbitrary infinite path in Γ^k. From (3.1) one can see that a marginal (formdary) measure μ_π on Ψ^π has the form

$$\mu_\pi(\omega_n)=p_{\omega(x_{-n})}\prod_{m=-n}^{n-1}p_{\omega(x_m)\omega(x_{m+1})}, \tag{4.7}$$

here $\omega_n:\{x_{-n},...,x_0,...,x_n\}\to\Psi$, i.e. ω_n is a configuration on $\{x_{-n},...,x_0,...,x_n\}$, p_i is an invariant vector for the matrix $(p_{ij})_{ij=1}^q$.

Using (2.3) and the form of h_* we have

$$|p_{ij}|_p=\frac{1}{\left|2\exp_p(J+2h)+(q-2)\exp_p(J)+q\sum_{i,j,1\neq j}\exp_p(h_*(i+j))\right|_p}=$$

$$=\frac{1}{\left|2(\exp_p(J+2h)-1)+2+(q-2)\exp_p(J)+q\sum_{i,j,i\neq j}\exp_p(h_*(i+j))\right|_p}\geq p. \tag{4.8}$$

for all i,j. Here we have used Lemma 2.2 and $|q|_p\leq \frac{1}{p}$. From (4.7) and (4.8) we find that μ_π is not bounded. Hence the theorem is proved. $\square$

Corollary 4.5. *The p-adic Gibbs measure μ corresponding to the p-adic Ising model on the Cayley tree of order k is bounded if $p\neq 2$, otherwise it is not bounded.*

Remark. From Theorems 4.2 and 4.4 we see that a phase transition occurs when p-adic Gibbs measures are not bounded. For the p-adic Ising model we know that a phase transition does not occur, so corollary 4.5 implies that if $p=2$ even in this case the p-adic Gibbs measure may not be bounded.

4.2 The uniqueness of Gibbs measure for the p-adic Potts model

If $q \in p\mathbb{N}$ then the equation (4.3) may have two solutions. But thus remains the case $q \notin p\mathbb{N}$ and a question naturally arises: is there a phase transition in this case or not? In this section we will prove the uniqueness of p-adic Gibbs measure for the p-adic Potts model in that case.

Let us first prove some technical results.

Lemma 4.6. *If $|a_i - 1|_p \le M$ and $|a_i|_p = 1$, $i = 1, 2, ..., n$, then*

$$\left| \prod_{i=1}^{n} a_i - 1 \right|_p \le M. \tag{4.9}$$

Proof. We prove this by induction on n. The case $n = 1$ is nothing but the condition of lemma. Suppose that (4.9) is valid for $n = m$. Now let $n = m + 1$. Then we have

$$\left| \prod_{i=1}^{m+1} a_i - 1 \right|_p = \left| \prod_{i=1}^{m+1} a_i - \prod_{i=1}^{m} a_i + \prod_{i=1}^{m} a_i - 1 \right|_p \le$$
$$\le \max\left\{ \left| \prod_{i=1}^{m} a_i (a_{m+1} - 1) \right|_p, \left| \prod_{i=1}^{m} a_i - 1 \right|_p \right\} \le M$$

This completes the proof. $\square$

Lemma 4.7. *Let $u_i = \prod_{j=1, j\neq i} \exp_p(h_j)$, where $h = (h_1, ..., h_{q-1})$, $\|h\|_p \le \frac{1}{p}$, then $|u_i|_p = 1$ and*

$$|u_i - 1|_p \le \frac{1}{p},$$

for all $i = 1, ..., q - 1$.

Proof. From Lemma 2.2 we infer that $|u_i|_p = 1$, since $|\exp_p(h_i) - 1|_p \le \frac{1}{p}$ and $|\exp_p(h_i)|_p = 1$. So all conditions of Lemma 4.6 are satisfied, hence we get $|u_i - 1|_p \le \frac{1}{p}$. $\square$

Write (see Theorem 3.3)

$$U_i(h, \theta, q) = \exp_p(F(h, \theta, q)) = \prod_{j=1, j\neq i}^{q-1} \frac{(\theta - 1)u_j + \sum_{j=1}^{q-1} u_j + 1}{\sum_{j=1}^{q-1} u_j + \theta}, \tag{4.10}$$

where $i = 1, ..., q - 1$. For brevity we use U_i instead of $U_i(h, \theta, q)$.

Lemma 4.8. *Let $q \notin p\mathbb{N}$, then $|U_i|_p = 1$ and*

$$|U_i - 1|_p \le \frac{1}{p}\|h\|_p \quad \textit{for} \quad i = 1, 2, ..., q - 1 \tag{4.11}$$

Proof. Put

$$K_i = \frac{(\theta-1)u_i + \sum_{j=1}^{q-1} u_j + 1}{\sum_{j=1}^{q-1} u_j + \theta}.$$

We compute the norm of K_i. From $|\theta-1|_p \leq \frac{1}{p}$, $|q|_p = 1$ and Lemma 4.7 we obtain

$$|K_i|_p = \left|\frac{(\theta-1)u_i + \sum_{j=1}^{q-1}(u_j-1) + q}{\sum_{j=1}^{q-1}(u_j-1) + (\theta-1) + q}\right|_p = 1.$$

Here we have used the strong triangle property of the norm $|\cdot|_p$. Observe that $U_i = \prod_{j=1, j\neq i}^{q-1} K_j$, and hence $|U_i|_p = 1$.

Now estimate $|K_i - 1|_p$:

$$|K_i - 1|_p = \left|\frac{(\theta-1)(u_i-1)}{\sum_{j=1}^{q-1}(u_j-1) + (\theta-1) + q}\right|_p \leq \frac{1}{p}\|h\|_p.$$

Consequently, we find that the conditions of Lemma 4.6 are satisfied for K_i, $i = 1, 2, ..., q-1$, whence (4.11).The lemma is proved.

Wtite

$$R_i(h_x, \theta, q) = \prod_{y\in S(x)} U_i(h_y, \theta, q), \quad x \in V. \tag{4.12}$$

Using Lemmas 4.6 and 4.8 one proves the following

Lemma 4.9. *Let $q \notin p\mathbb{N}$, then*

$$|R_i(h_x, \theta, q) - 1|_p \leq \frac{1}{p} \max_{y\in S(x)} \|h_y\|_p.$$

Corollary 4.10. *Let $q \notin p\mathbb{N}$, then*

$$\|h_x\|_p \leq \frac{1}{p} \max_{y\in S(x)} \|h_y\|_p, \quad x \in V. \tag{4.13}$$

Proof. From (3.7) and (4.12) one can see that $\exp_p(h_i^x) = R_i(h_x, \theta, q)$. Hence using Lemma 2.2 and Lemma 4.9 we infer that

$$|h_i^x|_p = |R_i(h_x, \theta, q) - 1|_p \leq \frac{1}{p} \max_{y\in S(x)} \|h_y\|_p,$$

whence (4.13). $\square$

Now we can formulate the main result of this subsection.

Theorem 4.11. *Let $k \geq 1$ and $q \notin p\mathbb{N}$, $|J|_p \leq \frac{1}{p}$ and p be any prime number. Then for the p-adic Potts model (2.4) on the Cayley tree of order k there is no phase transition.*

Proof. To prove it enough to show that $\Lambda = \{h_x \equiv 0\}$. In order to do this we will show that for arbitrary $\varepsilon > 0$ and every $x \in V$ we have $\|h_x\|_p < \varepsilon$. Let $n_0 \in \mathbb{N}$ be such that $\frac{1}{p^{n_0}} < \varepsilon$. According to Corollary 4.10 we have

$$\|h_x\|_p \le \frac{1}{p}\|h_{x_{i_0}}\|_p \le$$
$$\le \frac{1}{p^2}\|h_{x_{i_0,i_1}}\|_p \le \cdots \le \frac{1}{p^{n_0-1}}\|h_{x_{i_0,\dots,i_{n_0-2}}}\|_p \le \frac{1}{p^{n_0}} < \varepsilon,$$

here $x_{i_0,\dots,i_n,j}$, $j = 1, 2, ..., k$ are direct successors of $x_{i_0,\dots,i_n}$, where $\|h_{x_{i_0,\dots,i_m}}\|_p = \max_{1\le j\le k}\{\|h_{x_{i_0,\dots,i_{m-1},j}}\|_p\}$. This completes the proof. □

Remark. If $q \notin p\mathbb{N}$ then theorem 4.4 says that the p-adic Gibbs measure corresponding to the Potts model is bounded, hence by Theorem 4.11 we see that in this case there is only bounded p-adic Gibbs measure for the p-adic Potts model.

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